

LINEAR PROGRAMMING

12.1 Overview

12.1.1 An Optimisation Problem A problem which seeks to maximise or minimise a function is called an optimisation problem. An optimisation problem may involve maximisation of profit, production etc or minimisation of cost, from available resources etc.

12.1.2 A Linear Programming Problem (LPP)

A linear programming problem deals with the optimisation (maximisation/minimisation) of a **linear function** of two variables (say x and y) known as **objective function** subject to the conditions that the variables are non-negative and satisfy a set of linear inequalities (called **linear constraints**). A linear programming problem is a special type of optimisation problem.

12.1.3 Objective Function Linear function $Z = ax + by$, where a and b are constants, which has to be maximised or minimised is called a linear objective function.

12.1.4 Decision Variables In the objective function $Z = ax + by$, x and y are called decision variables.

12.1.5 Constraints The linear inequalities or restrictions on the variables of an LPP are called **constraints**. The conditions $x \geq 0$, $y \geq 0$ are called non-negative constraints.

12.1.6 Feasible Region The common region determined by all the constraints including non-negative constraints $x \geq 0$, $y \geq 0$ of an LPP is called the feasible region for the problem.

12.1.7 Feasible Solutions Points within and on the boundary of the feasible region for an LPP represent feasible solutions.

12.1.8 Infeasible Solutions Any Point outside feasible region is called an infeasible solution.

12.1.9 Optimal (feasible) Solution Any point in the feasible region that gives the optimal value (maximum or minimum) of the objective function is called an optimal solution.

Following theorems are fundamental in solving LPPs.

12.1.10 Theorem 1 Let R be the feasible region (convex polygon) for an LPP and let $Z = ax + by$ be the objective function. When Z has an optimal value (maximum or minimum), where x and y are subject to constraints described by linear inequalities, this optimal value must occur at a corner point (vertex) of the feasible region.

Theorem 2 Let R be the feasible region for a LPP and let $Z = ax + by$ be the objective function. If R is **bounded**, then the objective function Z has both a maximum and a minimum value on R and each of these occur at a corner point of R .

If the feasible region R is **unbounded**, then a maximum or a minimum value of the objective function may or may not exist. However, if it exists, it must occur at a corner point of R .

12.1.11 Corner point method for solving a LPP

The method comprises of the following steps :

- (1) Find the feasible region of the LPP and determine its corner points (vertices) either by inspection or by solving the two equations of the lines intersecting at that point.
- (2) Evaluate the objective function $Z = ax + by$ at each corner point.
Let M and m , respectively denote the largest and the smallest values of Z .
- (3) (i) When the feasible region is *bounded*, M and m are, respectively, the maximum and minimum values of Z .
(ii) In case, the feasible region is *unbounded*.
(a) M is the maximum value of Z , if the open half plane determined by $ax + by > M$ has no point in common with the feasible region. Otherwise, Z has no maximum value.
(b) Similarly, m is the minimum of Z , if the open half plane determined by $ax + by < m$ has no point in common with the feasible region. Otherwise, Z has no minimum value.

12.1.12 Multiple optimal points If two corner points of the feasible region are optimal solutions of the same type, i.e., both produce the same maximum or minimum, then any point on the line segment joining these two points is also an optimal solution of the same type.

12.2 Solved Examples

Short Answer (S.A.)

Example 1 Determine the maximum value of $Z = 4x + 3y$ if the feasible region for an LPP is shown in Fig. 12.1.

Solution The feasible region is bounded. Therefore, maximum of Z must occur at the corner point of the feasible region (Fig. 12.1).

Corner Point	Value of Z
O, (0, 0)	$4(0) + 3(0) = 0$
A (25, 0)	$4(25) + 3(0) = 100$
B (16, 16)	$4(16) + 3(16) = \mathbf{112}$ ← (Maximum)
C (0, 24)	$4(0) + 3(24) = 72$

Hence, the maximum value of Z is 112.

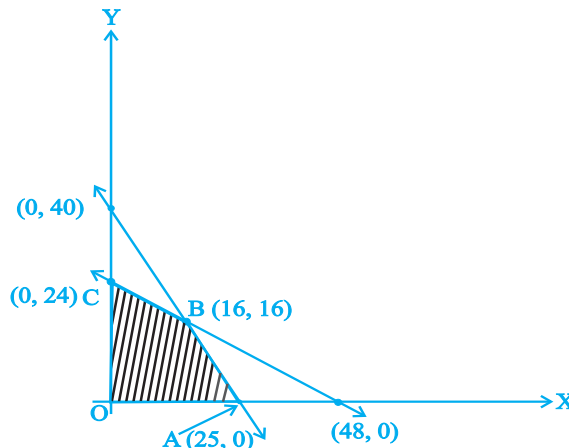


Fig.12.1

Fig. 12.1

Example 2 Determine the minimum value of $Z = 3x + 2y$ (if any), if the feasible region for an LPP is shown in Fig.12.2.

Solution The feasible region (R) is unbounded. Therefore minimum of Z may or may not exist. If it exists, it will be at the corner point (Fig.12.2).

Corner Point	Value of Z
A, (12, 0)	$3(12) + 2(0) = 36$
B (4, 2)	$3(4) + 2(2) = 16$
C (1, 5)	$3(1) + 2(5) = \mathbf{13}$ ← (smallest)
D (0, 10)	$3(0) + 2(10) = 20$

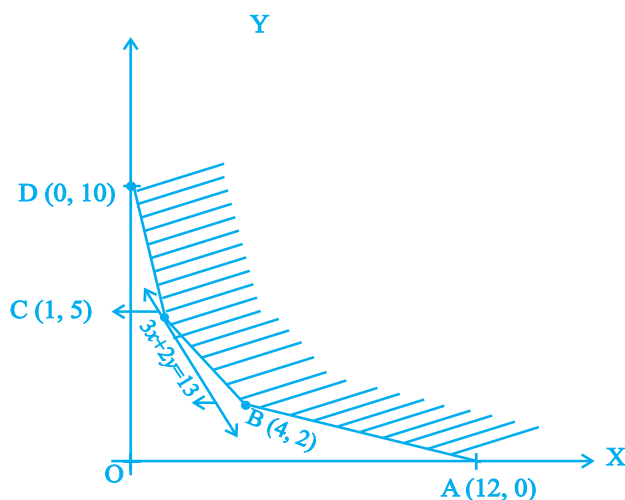


Fig. 12.2

Let us graph $3x + 2y < 13$. We see that the open half plane determined by $3x + 2y < 13$ and R do not have a common point. So, the smallest value 13 is the minimum value of Z .

Example 3 Solve the following LPP graphically:

Maximise $Z = 2x + 3y$,

subject to $x + y \leq 4$, $x \geq 0$, $y \geq 0$

Solution The shaded region (OAB) in the Fig. 12.3 is the feasible region determined by the system of constraints $x \geq 0$, $y \geq 0$ and $x + y \leq 4$.

The feasible region OAB is **bounded**, so, maximum value will occur at a corner point of the feasible region.

Corner Points are $O(0, 0)$, $A(4, 0)$ and $B(0, 4)$.

Evaluate Z at each of these corner point.

Corner Point	Value of Z
$O, (0, 0)$	$2(0) + 3(0) = 0$
$A(4, 0)$	$2(4) + 3(0) = 8$
$B(0, 4)$	$2(0) + 3(4) = 12$

← Maximum

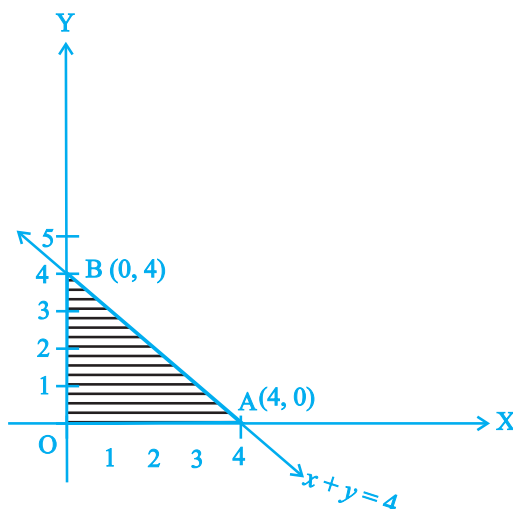


Fig. 12.3

Hence, the maximum value of Z is 12 at the point $(0, 4)$

Example 4 A manufacturing company makes two types of television sets; one is black and white and the other is colour. The company has resources to make at most 300 sets a week. It takes Rs 1800 to make a black and white set and Rs 2700 to make a coloured set. The company can spend not more than Rs 648000 a week to make television sets. If it makes a profit of Rs 510 per black and white set and Rs 675 per coloured set, how many sets of each type should be produced so that the company has maximum profit? Formulate this problem as a LPP given that the objective is to maximise the profit.

Solution Let x and y denote, respectively, the number of black and white sets and coloured sets made each week. Thus

$$x \geq 0, y \geq 0$$

Since the company can make at most 300 sets a week, therefore,

$$x + y \leq 300$$

Weekly cost (in Rs) of manufacturing the set is

$$1800x + 2700y$$

and the company can spend upto Rs. 648000. Therefore,

$$1800x + 2700y \leq 648000, \text{ i.e., or } 2x + 3y \leq 720$$

The total profit on x black and white sets and y colour sets is Rs $(510x + 675y)$. Let $Z = 510x + 675y$. This is the **objective function**.

Thus, the mathematical formulation of the problem is

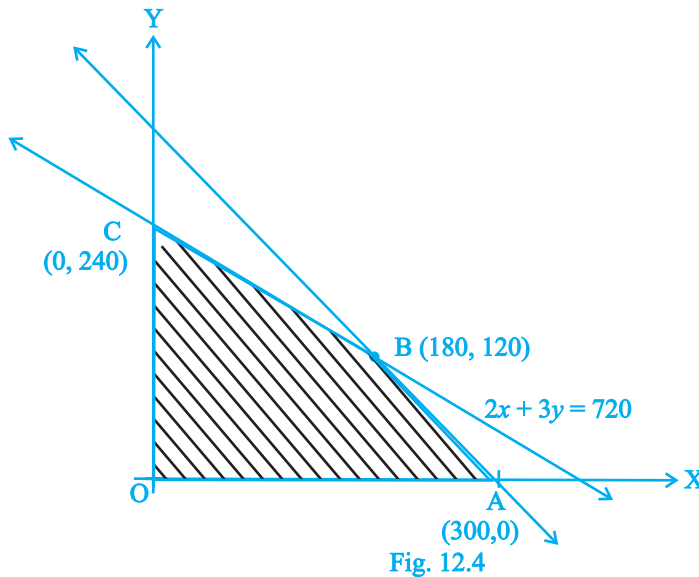
Maximise $Z = 510x + 675y$

subject to the constraints :

$$x + y = 300$$

$$2x + 3y = 720$$

$$x \geq 0, y \geq 0$$



Long Answer (L.A.)

Example 5 Refer to Example 4. Solve the LPP.

Solution The problem is :

Maximise $Z = 510x + 675y$

subject to the constraints :

$$x + y = 300$$

$$2x + 3y = 720$$

$$x \geq 0, y \geq 0$$

The feasible region OABC is shown in the Fig. 12.4.

Since the feasible region is bounded, therefore maximum of Z must occur at the corner point of OBC.

Corner Point	Value of Z	
O (0, 0)	$510 (0) + 675 (0) = 0$	
A (300, 0)	$510 (300) + 675 (0) = 153000$	
B (180, 120)	$510 (180) + 675 (120) = \mathbf{172800}$	← Maximum
C (0, 240)	$510 (0) + 675 (240) = 162000$	

Thus, maximum Z is 172800 at the point (180, 120), i.e., the company should produce 180 black and white television sets and 120 coloured television sets to get maximum profit.

Example 6 Minimise $Z = 3x + 5y$ subject to the constraints :

$$x + 2y \geq 10$$

$$x + y \geq 6$$

$$3x + y \geq 8$$

$$x, y \geq 0$$

Solution We first draw the graphs of $x + 2y = 10$, $x + y = 6$, $3x + y = 8$. The shaded region ABCD is the feasible region (R) determined by the above constraints. The feasible region is unbounded. Therefore, minimum of Z may or may not occur. If it occurs, it will be on the corner point.

Corner Point	Value of Z	
A (0, 8)	40	
B (1, 5)	28	
C (2, 4)	26	← smallest
D (10, 0)	30	



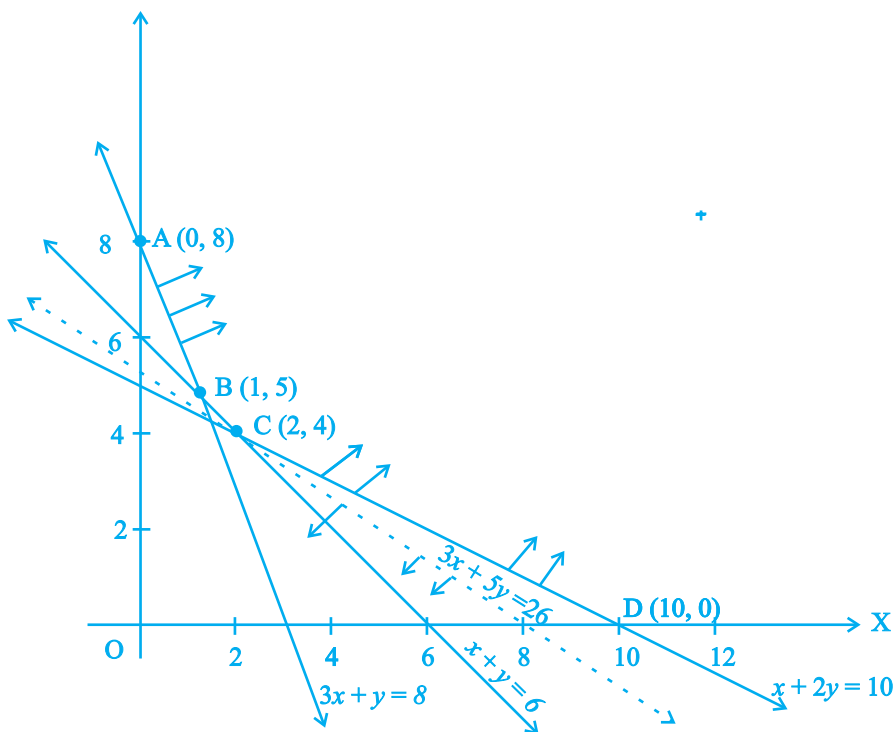


Fig. 12.5

Let us draw the graph of $3x + 5y < 26$ as shown in Fig. 12.5 by dotted line.

We see that the open half plane determined by $3x + 5y < 26$ and R do not have a point in common. Thus, 26 is the minimum value of Z.

Objective Type Questions

Choose the correct answer from the given four options in each of the Examples 7 to 8.

Example 7 The corner points of the feasible region determined by the system of linear constraints are $(0, 10)$, $(5, 5)$, $(15, 15)$, $(0, 20)$. Let $Z = px + qy$, where $p, q > 0$. Condition on p and q so that the maximum of Z occurs at both the points $(15, 15)$ and $(0, 20)$ is

- (A) $p = q$ (B) $p = 2q$ (C) $q = 2p$ (D) $q = 3p$

Solution The correct answer is (D). Since Z occurs maximum at $(15, 15)$ and $(0, 20)$, therefore, $15p + 15q = 0.p + 20q \Rightarrow q = 3p$.

Example 8 Feasible region (shaded) for a LPP is shown in the Fig. 14.6. Minimum of $Z = 4x + 3y$ occurs at the point

- (A) $(0, 8)$ (B) $(2, 5)$
(C) $(4, 3)$ (D) $(9, 0)$

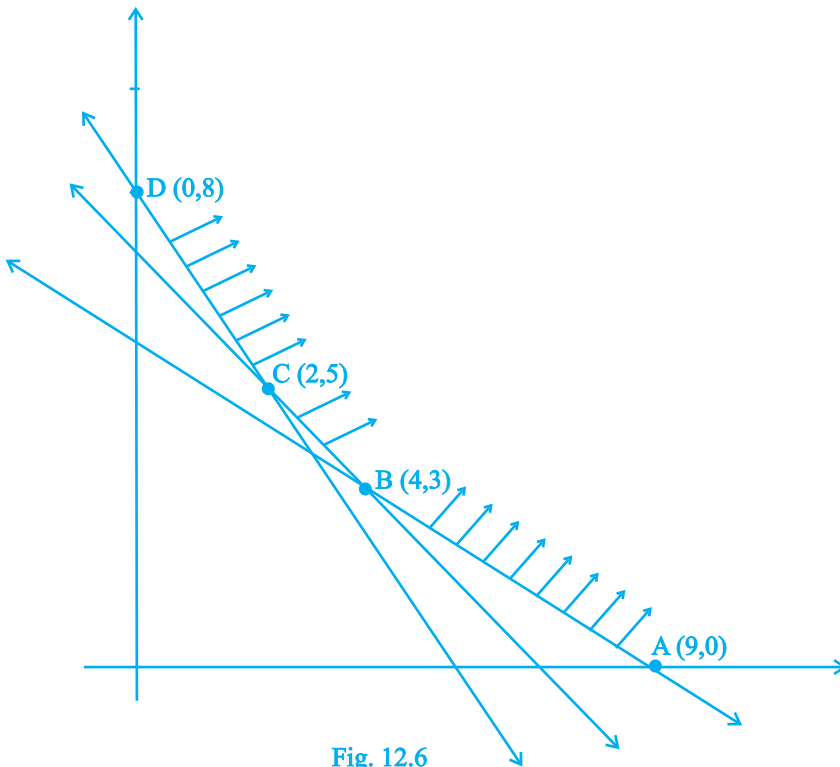


Fig. 12.6

Solution The correct answer is (B).

Fill in the blanks in each of the Examples 9 and 10:

Example 9 In a LPP, the linear function which has to be maximised or minimised is called a linear _____ function.

Solution Objective.

Example 10 The common region determined by all the linear constraints of a LPP is called the _____ region.

Solution Feasible.

State whether the statements in Examples 11 and 12 are **True** or **False**.

Example 11 If the feasible region for a linear programming problem is bounded, then the objective function $Z = ax + by$ has both a maximum and a minimum value on R.

Solution True

Example 12 The minimum value of the objective function $Z = ax + by$ in a linear programming problem always occurs at only one corner point of the feasible region.

Solution False

The minimum value can also occur at more than one corner points of the feasible region.

12.3 EXERCISE

Short Answer (S.A.)

1. Determine the maximum value of $Z = 11x + 7y$ subject to the constraints :
 $2x + y \leq 6, x \leq 2, x \geq 0, y \geq 0$.
2. Maximise $Z = 3x + 4y$, subject to the constraints: $x + y \leq 1, x \geq 0, y \geq 0$.
3. Maximise the function $Z = 11x + 7y$, subject to the constraints: $x \leq 3, y \leq 2, x \geq 0, y \geq 0$.
4. Minimise $Z = 13x - 15y$ subject to the constraints : $x + y \leq 7, 2x - 3y + 6 \geq 0, x \geq 0, y \geq 0$.
5. Determine the maximum value of $Z = 3x + 4y$ if the feasible region (shaded) for a LPP is shown in Fig.12.7.

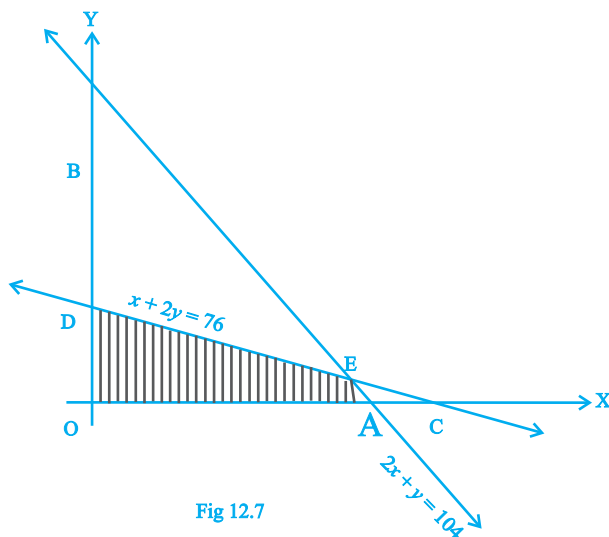


Fig 12.7

6. Feasible region (shaded) for a LPP is shown in Fig. 12.8.

Maximise $Z = 5x + 7y$.

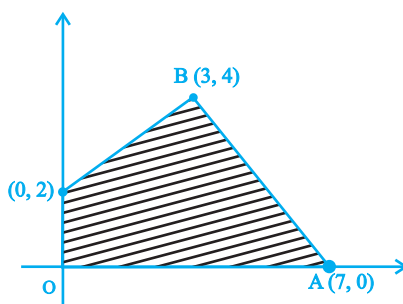
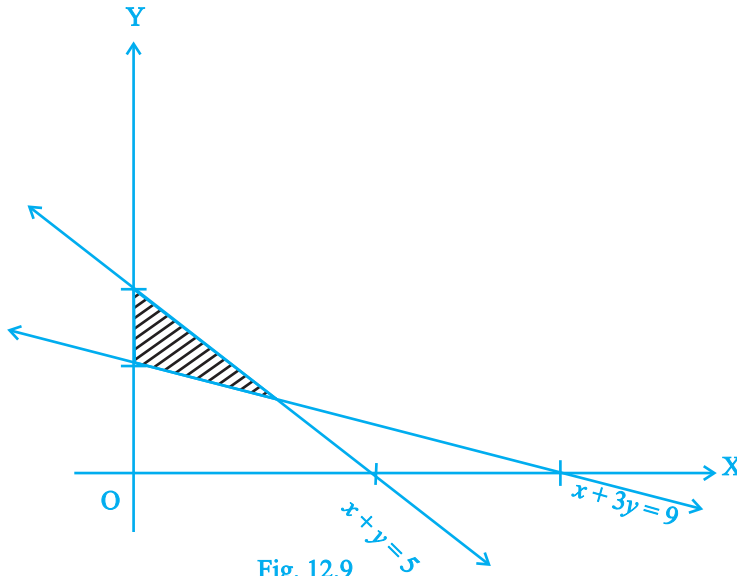
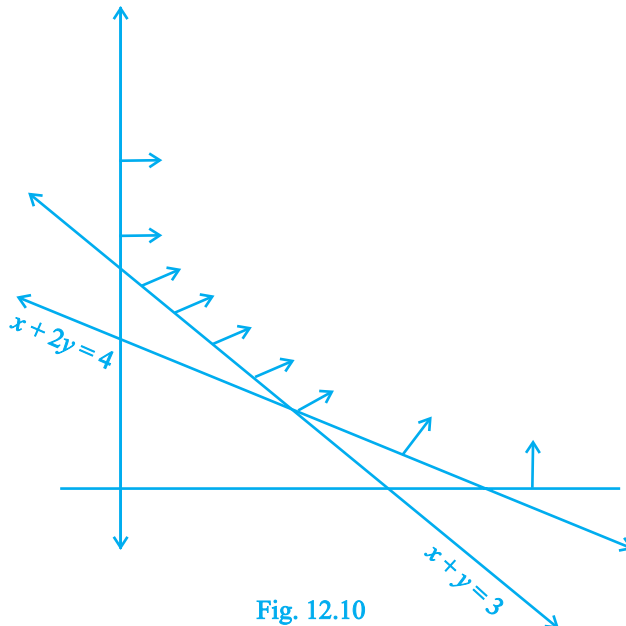


Fig. 12.8

7. The feasible region for a LPP is shown in Fig. 12.9. Find the minimum value of $Z = 11x + 7y$.



8. Refer to Exercise 7 above. Find the maximum value of Z .
9. The feasible region for a LPP is shown in Fig. 12.10. Evaluate $Z = 4x + y$ at each of the corner points of this region. Find the minimum value of Z , if it exists.



10. In Fig. 12.11, the feasible region (shaded) for a LPP is shown. Determine the maximum and minimum value of $Z = x + 2y$

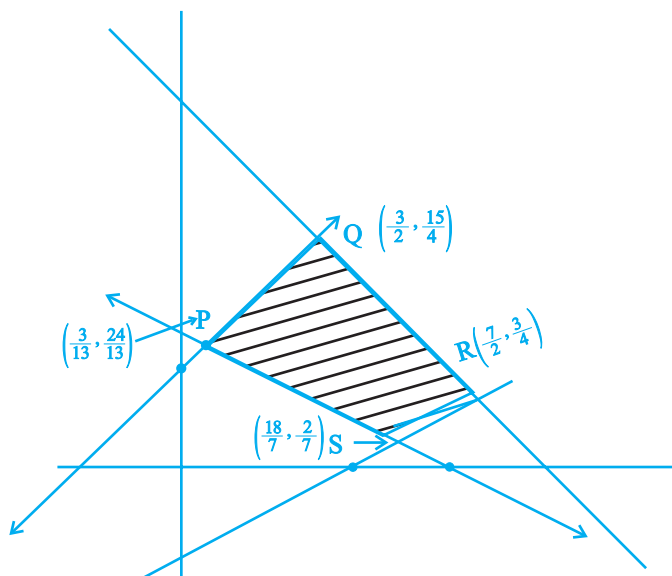


Fig. 12.11

11. A manufacturer of electronic circuits has a stock of 200 resistors, 120 transistors and 150 capacitors and is required to produce two types of circuits A and B. Type A requires 20 resistors, 10 transistors and 10 capacitors. Type B requires 10 resistors, 20 transistors and 30 capacitors. If the profit on type A circuit is Rs 50 and that on type B circuit is Rs 60, formulate this problem as a LPP so that the manufacturer can maximise his profit.
12. A firm has to transport 1200 packages using large vans which can carry 200 packages each and small vans which can take 80 packages each. The cost for engaging each large van is Rs 400 and each small van is Rs 200. Not more than Rs 3000 is to be spent on the job and the number of large vans can not exceed the number of small vans. Formulate this problem as a LPP given that the objective is to minimise cost.
13. A company manufactures two types of screws A and B. All the screws have to pass through a threading machine and a slotting machine. A box of Type A screws requires 2 minutes on the threading machine and 3 minutes on the slotting machine. A box of type B screws requires 8 minutes of threading on the threading machine and 2 minutes on the slotting machine. In a week, each machine is available for 60 hours.

On selling these screws, the company gets a profit of Rs 100 per box on type A screws and Rs 170 per box on type B screws.

Formulate this problem as a LPP given that the objective is to maximise profit.

- 14.** A company manufactures two types of sweaters : type A and type B. It costs Rs 360 to make a type A sweater and Rs 120 to make a type B sweater. The company can make at most 300 sweaters and spend at most Rs 72000 a day. The number of sweaters of type B cannot exceed the number of sweaters of type A by more than 100. The company makes a profit of Rs 200 for each sweater of type A and Rs 120 for every sweater of type B.

Formulate this problem as a LPP to maximise the profit to the company.

- 15.** A man rides his motorcycle at the speed of 50 km/hour. He has to spend Rs 2 per km on petrol. If he rides it at a faster speed of 80 km/hour, the petrol cost increases to Rs 3 per km. He has atmost Rs 120 to spend on petrol and one hour's time. He wishes to find the maximum distance that he can travel.

Express this problem as a linear programming problem.

Long Answer (L.A.)

- 16.** Refer to Exercise 11. How many of circuits of Type A and of Type B, should be produced by the manufacturer so as to maximise his profit? Determine the maximum profit.
- 17.** Refer to Exercise 12. What will be the minimum cost?
- 18.** Refer to Exercise 13. Solve the linear programming problem and determine the maximum profit to the manufacturer.
- 19.** Refer to Exercise 14. How many sweaters of each type should the company make in a day to get a maximum profit? What is the maximum profit.
- 20.** Refer to Exercise 15. Determine the maximum distance that the man can travel.
- 21.** Maximise $Z = x + y$ subject to $x + 4y \leq 8$, $2x + 3y \leq 12$, $3x + y \leq 9$, $x \geq 0$, $y \geq 0$.
- 22.** A manufacturer produces two Models of bikes - Model X and Model Y. Model X takes a 6 man-hours to make per unit, while Model Y takes 10 man-hours per unit. There is a total of 450 man-hour available per week. Handling and Marketing costs are Rs 2000 and Rs 1000 per unit for Models X and Y respectively. The total funds available for these purposes are Rs 80,000 per week. Profits per unit for Models X and Y are Rs 1000 and Rs 500, respectively. How many bikes of each model should the manufacturer produce so as to yield a maximum profit? Find the maximum profit.

23. In order to supplement daily diet, a person wishes to take some X and some Y tablets. The contents of iron, calcium and vitamins in X and Y (in milligrams per tablet) are given as below:

Tablets	Iron	Calcium	Vitamin
X	6	3	2
Y	2	3	4

The person needs at least 18 milligrams of iron, 21 milligrams of calcium and 16 milligram of vitamins. The price of each tablet of X and Y is Rs 2 and Re 1 respectively. How many tablets of each should the person take in order to satisfy the above requirement at the minimum cost?

24. A company makes 3 model of calculators: A, B and C at factory I and factory II. The company has orders for at least 6400 calculators of model A, 4000 calculator of model B and 4800 calculator of model C. At factory I, 50 calculators of model A, 50 of model B and 30 of model C are made every day; at factory II, 40 calculators of model A, 20 of model B and 40 of model C are made every day. It costs Rs 12000 and Rs 15000 each day to operate factory I and II, respectively. Find the number of days each factory should operate to minimise the operating costs and still meet the demand.

25. Maximise and Minimise $Z = 3x - 4y$
 subject to
- $$\begin{aligned} x - 2y &\leq 0 \\ -3x + y &\leq 4 \\ x - y &\leq 6 \\ x, y &\geq 0 \end{aligned}$$

Objective Type Questions

Choose the correct answer from the given four options in each of the Exercises 26 to 34.

26. The corner points of the feasible region determined by the system of linear constraints are (0, 0), (0, 40), (20, 40), (60, 20), (60, 0). The objective function is $Z = 4x + 3y$.

Compare the quantity in Column A and Column B

Column A

Maximum of Z

Column B

325

- (A) The quantity in column A is greater
 (B) The quantity in column B is greater



- (C) The two quantities are equal
 (D) The relationship can not be determined on the basis of the information supplied

27. The feasible solution for a LPP is shown in Fig. 12.12. Let $Z = 3x - 4y$ be the

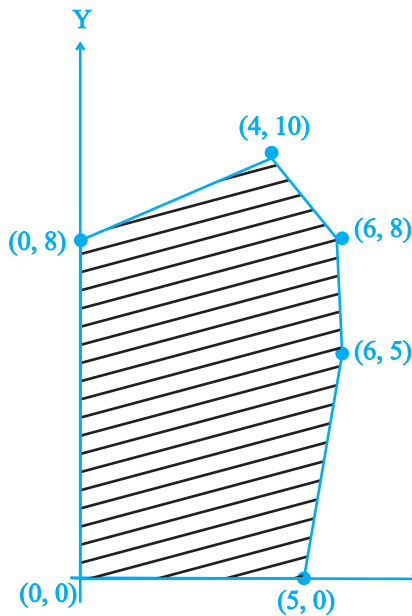
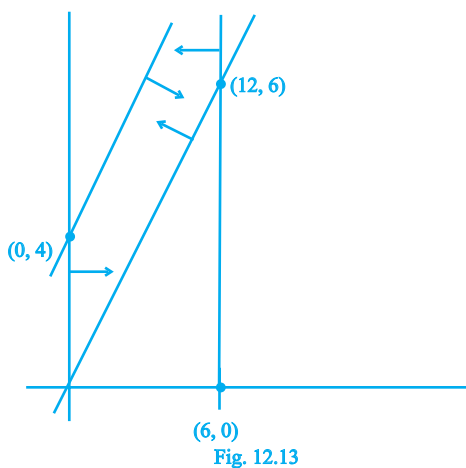


Fig. 12.12

objective function. Minimum of Z occurs at

- (A) (0, 0) (B) (0, 8) (C) (5, 0) (D) (4, 10)
28. Refer to Exercise 27. Maximum of Z occurs at
 (A) (5, 0) (B) (6, 5) (C) (6, 8) (D) (4, 10)
29. Refer to Exercise 27. (Maximum value of Z + Minimum value of Z) is equal to
 (A) 13 (B) 1 (C) -13 (D) -17
30. The feasible region for an LPP is shown in the Fig. 12.13. Let $F = 3x - 4y$ be the objective function. Maximum value of F is.



- (A) 0 (B) 8 (C) 12 (D) -18

31. Refer to Exercise 30. Minimum value of F is

- (A) 0 (B) -16 (C) 12 (D) does not exist

32. Corner points of the feasible region for an LPP are $(0, 2)$, $(3, 0)$, $(6, 0)$, $(6, 8)$ and $(0, 5)$.

Let $F = 4x + 6y$ be the objective function.

The Minimum value of F occurs at

- (A) $(0, 2)$ only
 (B) $(3, 0)$ only
 (C) the mid point of the line segment joining the points $(0, 2)$ and $(3, 0)$ only
 (D) any point on the line segment joining the points $(0, 2)$ and $(3, 0)$.

33. Refer to Exercise 32, Maximum of F - Minimum of F =

- (A) 60 (B) 48 (C) 42 (D) 18

34. Corner points of the feasible region determined by the system of linear constraints are $(0, 3)$, $(1, 1)$ and $(3, 0)$. Let $Z = px + qy$, where $p, q > 0$. Condition on p and q so that the minimum of Z occurs at $(3, 0)$ and $(1, 1)$ is

- (A) $p = 2q$ (B) $p = \frac{q}{2}$ (C) $p = 3q$ (D) $p = q$



Fill in the blanks in each of the Exercises 35 to 41.

35. In a LPP, the linear inequalities or restrictions on the variables are called _____.
36. In a LPP, the objective function is always _____
37. If the feasible region for a LPP is _____, then the optimal value of the objective function $Z = ax + by$ may or may not exist.
38. In a LPP if the objective function $Z = ax + by$ has the same maximum value on two corner points of the feasible region, then every point on the line segment joining these two points give the same _____ value.
39. A feasible region of a system of linear inequalities is said to be _____ if it can be enclosed within a circle.
40. A corner point of a feasible region is a point in the region which is the _____ of two boundary lines.
41. The feasible region for an LPP is always a _____ polygon.

State whether the statements in Exercises 42 to 45 are True or False.

42. If the feasible region for a LPP is unbounded, maximum or minimum of the objective function $Z = ax + by$ may or may not exist.
43. Maximum value of the objective function $Z = ax + by$ in a LPP always occurs at only one corner point of the feasible region.
44. In a LPP, the minimum value of the objective function $Z = ax + by$ is always 0 if origin is one of the corner point of the feasible region.
45. In a LPP, the maximum value of the objective function $Z = ax + by$ is always finite.

